Matrix Structural Analysis of Substructures

J. S. PRZEMIENIECKI*
Air Force Institute of Technology

Summary

A matrix method of linear structural analysis is described for the calculation of stresses and deflections in an aircraft structure divided into a number of structural components (substructures). The necessity for dividing a structure into these components arises either from the requirement that different types of analysis have to be used on different components, or because the capacity of the digital computer is not adequate to cope with the analysis of the complete structure. In the present method each substructure is first analyzed separately, assuming that all common boundaries (joints) with the adjacent substructures are completely fixed; these boundaries are then relaxed simultaneously and the actual boundary displacements are determined from the equations of equilibrium of forces at the boundary joints. The substructures are then analyzed separately again under the action of specified external loading and the previously determined boundary displacements.

Symbols

$\mathbf{u}^{(i)}$	= displacements on ith element
$\mathbf{u}$	= $\{\mathbf{u}^{(1)}\mathbf{u}^{(2)} \ldots \mathbf{u}^{(i)} \ldots\}$, element displacements
$\mathbf{U}$	= $\{\mathbf{U}_b\ \mathbf{U}_i\}$, structure displacements
$\mathbf{U}_b$	= boundary displacements
$\mathbf{U}_i$	= interior displacements
$\mathbf{U}^{(r)}$	= $\{\mathbf{u}_b^{(r)}\ \mathbf{u}_i^{(r)}\}$, substructure displacements
$\mathbf{u}_b^{(r)}$	= $\{\mathbf{v}^{(r)}\ \mathbf{w}^{(r)}\}$, boundary displacements for rth substructure
$\mathbf{u}_i^{(r)}$	= interior displacements for rth substructure
$\mathbf{v}^{(r)}$	= boundary displacements, other than boundary datum displacements
$\mathbf{v}_0^{(r)}$	= boundary displacements $\mathbf{v}^{(r)}$ measured relative to the boundary datum
$\mathbf{w}^{(r)}$	= $\{w_1^{(r)}\ w_2^{(r)}\ w_3^{(r)}\ w_4^{(r)}\ w_5^{(r)}\ w_6^{(r)}\}$, boundary datum displacements
$w_x\ w_y\ w_z$	= rigid body translations
$\theta_x\ \theta_y\ \theta_z$	= rigid body rotations
$\mathbf{P}$	= $\{\mathbf{P}_b\ \mathbf{P}_i\}$, externally applied forces
$\mathbf{P}_b$	= boundary forces
$\mathbf{P}_i$	= interior forces
$\mathbf{P}^{(r)}$	= $\{\mathbf{P}_b^{(r)}\ \mathbf{P}_i^{(r)}\}$, externally applied forces on rth substructure
$\mathbf{P}_b^{(r)}$	= boundary forces on rth substructure
$\mathbf{P}_i^{(r)}$	= interior forces on rth substructure
$\mathbf{Q}_b$	= resultant boundary forces for complete structure
$\mathbf{Q}_b^{(r)}$	= $\{\mathbf{Q}_v^{(r)}\ \mathbf{Q}_w^{(r)}\}$, boundary forces on rth substructure
$\mathbf{Q}_v^{(r)}$	= boundary forces corresponding to displacements $\mathbf{v}^{(r)}$
$\mathbf{Q}_w^{(r)}$	= boundary forces corresponding to displacements $\mathbf{w}^{(r)}$
$\mathbf{R}_b^{(r)}$	= boundary reactions due to $\mathbf{P}_i^{(r)}$
$\mathbf{s}^{(i)}$	= forces on ith element
$\mathbf{s}$	= $\{\mathbf{s}^{(1)}\ \mathbf{s}^{(2)} \ldots \mathbf{s}^{(i)} \ldots\}$, element forces
$\mathbf{s}^{(r)}$	= element forces on rth substructure
$\mathbf{k}^{(i)}$	= stiffness of ith element
$\mathbf{k}$	= $\lceil \mathbf{k}^{(1)}\ \mathbf{k}^{(2)} \ldots \mathbf{k}^{(i)} \ldots \rfloor$, element stiffnesses
$\mathbf{K}$	= stiffness matrix for complete structure
$\mathbf{K}_b^{(r)}$	= boundary stiffness of rth substructure
$\mathbf{K}_b$	= boundary stiffness for complete structure
$\mathbf{K}_{ii}\ \mathbf{K}_{ib}\ \mathbf{K}_{bi}\ \mathbf{K}_{bb}$	= component submatrices of $\mathbf{K}$, defined by Eq. (2)
$\mathbf{K}_{ii}^{(r)}\ \mathbf{K}_{ib}^{(r)}\ \mathbf{K}_{bi}^{(r)}\ \mathbf{K}_{bb}^{(r)}$	= component submatrices of $\mathbf{K}^{(r)}$, defined by Eq. (24)
$\mathbf{K}_{vv}^{(r)}\ \mathbf{K}_{vw}^{(r)}\ \mathbf{K}_{wv}^{(r)}\ \mathbf{K}_{ww}^{(r)}$	= component submatrices of $\mathbf{K}_b^{(r)}$, defined by Eq. (29a)
$\mathbf{F}_{vv}^{(r)}$	= flexibility of rth substructure relative to the substructure datum $w_1 = w_2 = \ldots = w_6 = 0$
$\mathbf{a}$	= transformation matrix defined by Eq. (19)
$\mathbf{a}_b$	= transformation matrix defined by Eq. (44)
$\mathbf{T}^{(r)}$	= transformation matrix for rth substructure, defined by Eqs. (31) and (59)
$\mathbf{T}_{vw}\ \mathbf{T}_{v\theta}$	= transformation matrices defined Eqs. (60) or (60a) and (61), respectively
$\mathbf{T}_{ww}\ \mathbf{T}_{w\theta}$	= transformation matrices defined by Eqs. (49) and (51)
$\mathbf{A}\ \mathbf{B}\ \mathbf{C}\ \mathbf{D}$	= submatrices defined by Eq. (51)
$\mathbf{I}$	= unit matrix
$x_j\ y_j\ z_j$	= coordinates of point j relative to structure datum
$x_k\ y_k\ z_k$	= coordinates of point k on the substructure boundary, relative to structure datum
$\bar{x}_j\ \bar{y}_j\ \bar{z}_j$	= coordinates of point j relative to the substructure datum point k, defined by Eq. (62)
n	= total number of boundary displacements on one substructure
λ_{ij}	= direction cosine of an angle between i and j directions
$[\ldots]$	= matrix
$\lceil \ldots \rfloor$	= diagonal matrix
$\{\ldots\}$	= column matrix

Superscripts

r	= rth substructure
i	= ith element
α	= substructure boundaries fixed
β	= correction due to boundary relaxation

Note: Boldface type is used to denote matrices.

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* Associate Professor, Dept. of Mechanics.

(1) Introduction

The fundamental problem in the elastic analysis of aircraft structures is the determination of the distribution of stresses and deflections under prescribed loads and constraints. For certain types of structures this problem can be tackled by a direct solution of the differential equations of elasticity describing the elastic behavior of the structure; the best example of such a solution is the Engineers Theory of Bending applied to box beam structures. These analytical solutions, however, are usually based on certain simplifying assumptions which are too restrictive particularly when applied to complex structures. This means that numerical methods must invariably be used in the analysis of aircraft structures to include the various structural effects which could not conveniently be accounted for in the analytical methods.

There are basically two types of numerical methods which can be used in structural analysis: either the differential equations describing the deflections and/or stresses in the structure are solved by numerical computing procedures or, alternatively, the structure is idealized into an assembly of discrete structural elements, having an assumed form of stress or displacement distribution, and the complete solution is then obtained by combining these individual, approximate stress or displacement distributions in a manner which satisfies the force equilibrium and displacement compatibility at the junctions of these elements. Methods based on the discrete element idealization have been used extensively for the analysis of aircraft structures.[1–11] All these methods involve appreciable quantities of linear algebra which must be organized into a systematic sequence of operations and to this end the use of matrix algebra is a convenient method of defining the various processes involved in the analysis without the necessity of writing out the complete operations in full. Furthermore, the formulation of a specific method of analysis in matrix algebra is ideally suited for subsequent solution on the digital computer and it also allows for an easy and systematic compilation of the required data.

Matrix methods of linear structural analysis based on discrete element idealization may be classified broadly into two groups:

(1) Displacement methods (stiffness methods) in which geometrically compatible states in individual elements are combined to give equilibrium, and

(2) Force methods (flexibility methods) in which equilibrium states in individual elements are combined to give geometrical compatibility.

A combined force/displacement method was proposed by Klein[12] where the equations of equilibrium and force displacement relations for all elements are solved jointly for the forces and displacements. This method, however, is of somewhat different type; it is based on the numerical integration of the differential equations of elasticity. For a comprehensive summary of the numerous publications in the field of matrix structural analysis Refs. 13 and 14 may be consulted.

One of the characteristic features of the displacement methods is that the question of statical redundancy does not arise there, as the problem is directed toward the solution for unknown displacements at the joints, and the resulting stress distribution is calculated subsequently from the displacements. In these terms there are always as many equations of equilibrium available as there are unknowns. In the force methods, on the other hand, the degree of redundancy for the idealized structure must be determined, since the problem is directed toward the solution for redundant forces (or groups of forces). The equations of equilibrium in terms of forces are inadequate in number to determine all the internal forces and they must therefore be supplemented by the equations of deflection compatibility. Because of the large number of unknown displacements involved, the application of the Displacement Method to complex aircraft structures had to await the introduction of large capacity digital computers before it could compete on equal terms with the Force Methods which generally require a smaller number of unknowns.

At the present time both methods are used extensively for the analysis of complex structures. The choice of the method of analysis depends mainly on the type of structure considered, but it may also be influenced by the availability of some routine computing program for a particular structure. This implies that, in general, different structural components may have to be analyzed by different methods. Thus, the complete structure would have to be considered as consisting of a number of distinct regions, which will be referred to as *substructures*, where different methods of analysis may be used. Another reason why the concept of substructures may be advantageous is that the number of unknowns, particularly when the displacement method is used, very often exceeds the capacity of the digital computer and the solution based on structural partitioning may be necessary to get any results at all with the available computing facility.

In the substructure method of analysis the complete structure is initially partitioned into a number of substructures, whose boundaries are specified arbitrarily, including even the possibility of one substructure being inside another. However, it should be emphasized that the method of partitioning affects subsequent matrix operations and special care must be exercised to ensure that the substructure boundaries are selected in the most economical way; the substructure selection and other related problems will be discussed later. To combine the individual substructure solutions structural coupling between adjacent substructures can be introduced either by special interaction redundant systems (groups of redundant forces) or simply by matching the corresponding displacements on the boundaries. The substructure analysis based on interaction redundant systems has been described by Argyris and Kelsey in

Ref. 6, where the possibility of matching the boundary displacements was also mentioned.

The object of this paper is to present the elastic analysis of large structures using the substructure concept with the displacements as unknowns in the analysis. The proposed Substructure Method of analysis[15] consists essentially of separate analyses of each substructure with all generalized displacements on common boundaries completely constrained, followed then by the relaxation of these boundaries to ensure equilibrium and the calculation of the substructure boundary displacements. Naturally, the solution for the boundary displacements involves a considerably smaller number of unknowns compared with the solution for the complete structure without partitioning. Each substructure can then be analyzed separately again under known substructure loading and boundary displacements; and this can be achieved without much difficulty since the substructures analyzed would be of a relatively small size.

The present analysis falls into a broad class of relaxation techniques, although it differs from the conventional techniques in one important aspect; only a single relaxation is used. The substructure method is somewhat analogous to the Hardy Cross[16] moment distribution method used very effectively in civil engineering structures. The essential difference, however, is that in the substructure method all fixed boundaries are relaxed simultaneously and the unknown boundary displacements are found from the solution of equilibrium equations at the joints. In the Hardy Cross method, on the other hand, one joint at a time is relaxed and the bending moments are modified to ensure equilibrium at the released joint. This process is repeated on all other joints and then the same sequence of releases repeated again and again until the desired degree of accuracy is achieved.

(2) General Theory

The complete set of equilibrium equations for the structure, regarded as a free body, may be written in matrix form as

$$\mathbf{K}\,\mathbf{U} = \mathbf{P} \tag{1}$$

where $\mathbf{K}$ is the stiffness matrix and $\mathbf{U}$ represents a column matrix of displacements corresponding to external forces $\mathbf{P}$. By suppressing a suitably chosen set of displacements to eliminate rigid body displacements the matrix $\mathbf{K}$ is rendered nonsingular and then Eq. (1) can be solved for the unknown displacements $\mathbf{U}$. In the following discussion the structure is divided into substructures by introducing interior boundaries. The column matrix of boundary displacements, common to two or more substructures, is denoted by $\mathbf{U}_b$, and the matrix of interior displacements (each of which occurs at an interior point of only one substructure) is $\mathbf{U}_i$. If the corresponding external forces are denoted by matrices $\mathbf{P}_b$ and $\mathbf{P}_i$, then Eq. (1) may be written in partitioned form as

$$\begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{U}_b \\ \mathbf{U}_i \end{bmatrix} = \begin{bmatrix} \mathbf{P}_b \\ \mathbf{P}_i \end{bmatrix} \tag{2}$$

It will now be assumed that the total displacements of the structure may be calculated from the superposition of two matrices such that

$$\mathbf{U} = \mathbf{U}^{(\alpha)} + \mathbf{U}^{(\beta)} \tag{3}$$

where $\mathbf{U}^{(\alpha)}$ denotes the column matrix of displacements due to $\mathbf{P}_i$ with $\mathbf{U}_b = \mathbf{0}$, while $\mathbf{U}^{(\beta)}$ represents the necessary corrections to the displacements $\mathbf{U}^{(\alpha)}$ to allow for boundary displacements $\mathbf{U}_b$ with $\mathbf{P}_i = \mathbf{0}$. Thus Eq. (3) may also be written as

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_b \\ \mathbf{U}_i \end{bmatrix} = \begin{bmatrix} \mathbf{U}_b^{(\alpha)} \\ \mathbf{U}_i^{(\alpha)} \end{bmatrix}_{\substack{\text{boundaries} \\ \text{fixed}}} + \begin{bmatrix} \mathbf{U}_b^{(\beta)} \\ \mathbf{U}_i^{(\beta)} \end{bmatrix}_{\substack{\text{correction due} \\ \text{to boundary} \\ \text{relaxation}}} \tag{3a}$$

where by definition

$$\mathbf{U}_b^{(\alpha)} = \mathbf{0} \tag{4}$$

Similarly, corresponding to the displacements $\mathbf{U}^{(\alpha)}$ and $\mathbf{U}^{(\beta)}$ the external forces $\mathbf{P}$ can be separated into

$$\mathbf{P} = \mathbf{P}^{(\alpha)} + \mathbf{P}^{(\beta)} \tag{5}$$

or

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_b \\ \mathbf{P}_i \end{bmatrix} = \begin{bmatrix} \mathbf{P}_b^{(\alpha)} \\ \mathbf{P}_i^{(\alpha)} \end{bmatrix} + \begin{bmatrix} \mathbf{P}_b^{(\beta)} \\ \mathbf{P}_i^{(\beta)} \end{bmatrix} \tag{5a}$$

where by definition

$$\mathbf{P}_i^{(\alpha)} = \mathbf{P}_i \tag{6}$$

and

$$\mathbf{P}_i^{(\beta)} = \mathbf{0} \tag{7}$$

When the substructure boundaries are fixed it can readily be shown, using Eq. (2), that

$$\mathbf{U}_i^{(\alpha)} = \mathbf{K}_{ii}^{-1}\,\mathbf{P}_i \tag{8}$$

and

$$\mathbf{P}_b^{(\alpha)} = \mathbf{K}_{bi}\,\mathbf{K}_{ii}^{-1}\,\mathbf{P}_i = \mathbf{R}_b \tag{9}$$

It should be noted that $\mathbf{P}_b^{(\alpha)}$ represents boundary reactions necessary to maintain $\mathbf{U}_b = \mathbf{0}$ when the interior forces $\mathbf{P}_i$ are applied. When the substructure boundaries are relaxed, the displacements $\mathbf{U}^{(\beta)}$ can be determined also from Eq. (2), so that

$$\mathbf{U}_i^{(\beta)} = -\mathbf{K}_{ii}^{-1}\,\mathbf{K}_{ib}\,\mathbf{U}_b^{(\beta)} \tag{10}$$

$$\mathbf{U}_b^{(\beta)} = \mathbf{K}_b^{-1}\,\mathbf{P}_b^{(\beta)} \tag{11}$$

where

$$\mathbf{K}_b = \mathbf{K}_{bb} - \mathbf{K}_{bi}\,\mathbf{K}_{ii}^{-1}\,\mathbf{K}_{ib} \tag{12}$$

represents the boundary stiffness matrix. The matrix

$\mathbf{P}_b^{(\beta)}$ can be determined from Eqs. (5a) and (9) and hence

$$\mathbf{P}_b^{(\beta)} = \mathbf{P}_b - \mathbf{P}_b^{(\alpha)} = \mathbf{P}_b - \mathbf{K}_{bi}\,\mathbf{K}_{ii}^{-1}\,\mathbf{P}_i = \mathbf{Q}_b \quad (13)$$

When the boundary displacements are set equal to zero, the substructures are completely isolated from one another so that application of an interior force causes displacements in only one substructure. It is therefore evident that the interior displacements $\mathbf{U}_i^{(\alpha)}$ with boundaries fixed can be calculated for each substructure separately, using Eq. (8). Although the determination of boundary displacements $\mathbf{U}_b^{(\beta)}$ involves the complete structure, it will be shown in later sections of the paper that the boundary stiffness matrix $\mathbf{K}_b$ may be calculated by superposition of component matrices. Considerable computational advantage is also derived from the fact that $\mathbf{K}_b$ is of much lower order than the complete stiffness matrix **K**.

(3) Matrix Displacement Method

Only the basic principles of the displacement method of analysis will be discussed briefly, leading to the formulation of Eq. (1); further details can be found in the numerous papers on this subject (see for example Refs. 3 and 4). The fundamental assumptions used in the analysis are that the structure can be satisfactorily represented by an assembly of discrete elements having simplified elastic properties and that these elements are interconnected so as to represent the actual, continuous structure. The deformations of the boundaries of adjacent elements are mutually compatible and the stresses within each element are equilibrated by a set of the element forces $\mathbf{s}^{(i)}$ in the directions of the element displacements $\mathbf{u}^{(i)}$. Furthermore, it is assumed that the element forces are related to the corresponding element displacements by an element stiffness matrix $\mathbf{k}^{(i)}$, and this can be expressed in matrix notation as

$$\mathbf{s}^{(i)} = \mathbf{k}^{(i)}\,\mathbf{u}^{(i)} \quad (14)$$

where the superscript (i) denotes the ith element. The forces and displacements on each element must refer to a common datum set of axes and this generally implies that if the element stiffness matrix is calculated first with respect to its own axes, then this matrix must be transformed to account for the actual location of the element with respect to the datum axes (Ox, Oy, Oz).

For the complete structure, Eqs. (14) can be combined into a single matrix equation

$$\mathbf{s} = \mathbf{k}\,\mathbf{u} \quad (15)$$

where

$$\mathbf{s} = \{\mathbf{s}^{(1)}\,\mathbf{s}^{(2)}\ldots\mathbf{s}^{(i)}\ldots\} \quad (16)$$

$$\mathbf{k} = \lceil\mathbf{k}^{(1)}\,\mathbf{k}^{(2)}\ldots\mathbf{k}^{(i)}\ldots\rfloor \quad (17)$$

$$\mathbf{u} = \{\mathbf{u}^{(1)}\,\mathbf{u}^{(2)}\ldots\mathbf{u}^{(i)}\ldots\} \quad (18)$$

The elements displacements **u** can be expressed in terms of the structure displacements **U** corresponding to the forces **P** by the equation

$$\mathbf{u} = \mathbf{a}\mathbf{U} \quad (19)$$

where **a** is a rectangular matrix in which every row consists of zeros except for a single term of unity, the position of which identifies that element of **u** which corresponds to the particular element of **U** (see Ref. 3, Vol. 27, p. 127). In other words, the function of the matrix **a** is to select the appropriate displacement from **U** and then place it in the required order, element by element, in the element displacement matrix **u**.

It can be shown, by the application of the Virtual Work theorem, that the equations of equilibrium at structural joints can be represented by

$$\mathbf{a}'\,\mathbf{s} = \mathbf{P} \quad (20)$$

where **P** is a column matrix of externally applied forces and the prime (′) is used to indicate matrix transposition. Substituting Eqs. (15) and (19) into Eq. (20), it follows that

$$\mathbf{a}'\mathbf{k}\,\mathbf{a}\mathbf{U} = \mathbf{P} \quad (21)$$

or

$$\mathbf{K}\,\mathbf{U} = \mathbf{P} \quad (21a)$$

where

$$\mathbf{K} = \mathbf{a}'\mathbf{k}\,\mathbf{a} \quad (22)$$

is referred to as the stiffness matrix for the complete structure regarded as a free body. In practice the matrix multiplication **a′k a** is never carried out since this multiplication is equivalent to the placing of matrix elements from $\mathbf{k}^{(i)}$ in their correct positions in the larger framework of the matrix **K** and then summing all the overlapping terms, which is a simpler operation than the calculation of the matrix product **a′k a** and it can be programmed directly for a digital computer without actual setting up of the transformation matrix **a**.

Eq. (1) represents equations of equilibrium at all the joints, including the reaction points, for a completely unconstrained structure. This implies that the loading matrix **P** constitutes a group of forces in static equilibrium. From the consideration of overall equilibrium of the structure, it is clear that there must be six dependent equations between the forces **P**, corresponding to the six rigid body degrees of freedom which are unconstrained, and this dependence must be eliminated from Eq. (1) in order to render the matrix **K** nonsingular. It is, therefore, necessary to restrain the rigid body degrees of freedom in Eq. (1) by assuming that six displacements* at certain selected points are equal to zero and eliminating the corresponding rows and columns from the complete stiffness matrix **K**. Only then would it be possible to obtain the solution for displacements from

* For simplicity rotations of joints (bending deformations) are not considered here.

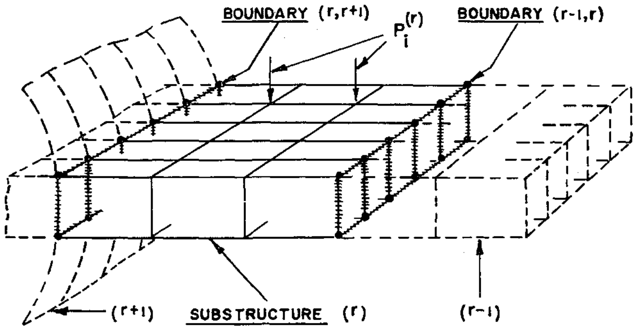

FIG. 1. Typical substructure with fixed boundaries.

$$\mathbf{U} = \mathbf{K}^{-1}\,\mathbf{P} \tag{23}$$

where all matrices are reduced in size to exclude forces at the selected zero displacements. The six zero displacements must be chosen in such a way as to ensure that rigid body degrees of freedom are completely restrained (3 translations and 3 rotations). Naturally, additional degrees of freedom (displacements) may also be restrained. For example, in the analysis of symmetrical structures under symmetrical loading the out-of-plane displacements at the plane of symmetry must be equal to zero. It must be emphasized, however, that it is important to ensure that the eliminated six displacements do in fact restrain all rigid body degrees of freedom. A simple method for checking this requirement is discussed in the Appendix.

(4) The Substructure Displacements and Forces: Boundaries Fixed

(4.1) Matrix Displacement Analysis

The stiffness matrix of the rth substructure, regarded as a free body, can conveniently be partitioned into

$$\mathbf{K}^{(r)} = \begin{bmatrix} \mathbf{K}_{bb}^{(r)} & \mathbf{K}_{bi}^{(r)} \\ \mathbf{K}_{ib}^{(r)} & \mathbf{K}_{ii}^{(r)} \end{bmatrix} \tag{24}$$

where the superscript r denotes the rth substructure and the subscripts b and i refer to the boundary and interior displacements, respectively. Naturally, because of symmetry of the stiffness matrix $\mathbf{K}_{bi}^{(r)}$ is a transpose of $\mathbf{K}_{ib}^{(r)}$—i.e., $\mathbf{K}_{bi}^{(r)} = (\mathbf{K}_{ib}^{(r)})'$. Using the above stiffness matrix, the substructure displacements $\mathbf{U}^{(r)}$ can be related to the external forces $\mathbf{P}^{(r)}$ by the following equation

$$\mathbf{K}^{(r)}\,\mathbf{U}^{(r)} = \mathbf{P}^{(r)} \tag{25}$$

or

$$\begin{bmatrix} \mathbf{K}_{bb}^{(r)} & \mathbf{K}_{bi}^{(r)} \\ \mathbf{K}_{ib}^{(r)} & \mathbf{K}_{ii}^{(r)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_b^{(r)} \\ \mathbf{u}_i^{(r)} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_b^{(r)} \\ \mathbf{P}_i^{(r)} \end{bmatrix} \tag{25a}$$

When the substructure boundaries on the complete structure are fixed, the boundary fixing must be sufficient to restrain rigid body degrees of freedom on each substructure considered separately. A typical substructure with fixed boundaries is shown in Fig. 1. The substructure interior displacements and boundary reactions due to $\mathbf{P}_i^{(r)}$ when $\mathbf{u}_b^{(r)} = \mathbf{0}$ can be determined from Eqs. (8) and (9) and therefore

$$[\mathbf{u}_i^{(r)}]_{\substack{\text{boundaries}\\ \text{fixed}}} = (\mathbf{K}_{ii}^{(r)})^{-1}\,\mathbf{P}_i^{(r)} \tag{26}$$

and

$$\mathbf{R}_b^{(r)} = \mathbf{K}_{bi}^{(r)}\,(\mathbf{K}_{ii}^{(r)})^{-1}\,\mathbf{P}_i^{(r)} \tag{27}$$

where the matrix inversion of $\mathbf{K}_{ii}^{(r)}$ is permissible because the boundary fixing restrains all rigid body degrees of freedom. Before considering "matching" of displacements on common boundaries, it is necessary to evaluate the substructure stiffnesses associated with the displacements $\mathbf{u}_b^{(r)}$. To determine these stiffnesses Eq. (12) is applied to the rth substructure and it follows immediately that

$$\mathbf{K}_b^{(r)} = \mathbf{K}_{bb}^{(r)} - \mathbf{K}_{bi}^{(r)}\,(\mathbf{K}_{ii}^{(r)})^{-1}\,\mathbf{K}_{ib}^{(r)} \tag{28}$$

which will be used subsequently to assembly the boundary stiffness matrix $\mathbf{K}_b$ for the complete structure.

(4.2) Matrix Force Analysis

Some substructures may be analyzed more conveniently by the matrix force method. For these cases the boundary reactions $\mathbf{R}_b^{(r)}$ due to $\mathbf{P}_i^{(r)}$ for fixed boundaries can be obtained from the general analysis and this presents no special difficulty. On the other hand, the calculation of the boundary stiffness $\mathbf{K}_b^{(r)}$, using the force method, is more difficult than in the case of the displacement method described in Section (4.1).

To evaluate the stiffness matrix $\mathbf{K}_b^{(r)}$ six zero displacements $\mathbf{w}^{(r)}$ are selected on the substructure boundary to restrain rigid body degrees of freedom and then unit loads are applied in the directions of the remaining $(n - 6)$ boundary displacements $\mathbf{v}^{(r)}$, where n is the total number of displacements on the boundary. The solution for displacements gives then the flexibility matrix $\mathbf{F}_{vv}^{(r)}$, relative to the fixed datum based on the selected six zero displacements, which will be used to determine the stiffness matrix $\mathbf{K}_b^{(r)}$.

The boundary forces and displacements are related by the equation

$$\mathbf{K}_b^{(r)}\,\mathbf{u}_b^{(r)} = \mathbf{Q}_b^{(r)} \tag{29}$$

or

$$\begin{bmatrix} \mathbf{K}_{vv}^{(r)} & \mathbf{K}_{vw}^{(r)} \\ \mathbf{K}_{wv}^{(r)} & \mathbf{K}_{ww}^{(r)} \end{bmatrix} \begin{bmatrix} \mathbf{v}^{(r)} \\ \mathbf{w}^{(r)} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_v^{(r)} \\ \mathbf{Q}_w^{(r)} \end{bmatrix} \tag{29a}$$

where $\mathbf{v}^{(r)}$ represents the $(\mathbf{n} - 6)$ displacements, $\mathbf{w}^{(r)}$ represents the six datum displacements, while $\mathbf{Q}_v^{(r)}$ and $\mathbf{Q}_w^{(r)}$ are the corresponding applied forces and

$$\mathbf{K}_{vv}^{(r)} = (\mathbf{F}_{vv}^{(r)})^{-1} \tag{30}$$

The column matrix for the boundary displacements $\mathbf{u}_b^{(r)}$ can be expressed as

$$\mathbf{u}_b^{(r)} = \begin{bmatrix} \mathbf{v}^{(r)} \\ \mathbf{w}^{(r)} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0^{(r)} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{T}^{(r)} \\ \mathbf{I} \end{bmatrix} \mathbf{w}^{(r)} \tag{31}$$

where $\mathbf{v}_0^{(r)}$ are the displacements $\mathbf{v}^{(r)}$ measured relative to the fixed datum and $\mathbf{T}^{(r)}$ is a transformation matrix derived in the Appendix. The derivation of this transformation matrix requires that three of the displacements in $\mathbf{w}^{(r)}$, say $w_1^{(r)}$, $w_2^{(r)}$, and $w_3^{(r)}$, must refer to three translations of a point (joint) on the substructure boundary. Applying now virtual displacements $\delta\mathbf{w}^{(r)}$, it follows from Eq. (31) that

$$\delta\mathbf{v}^{(r)} = \mathbf{T}^{(r)}\,\delta\mathbf{w}^{(r)} \tag{32}$$

From the Principle of Virtual Work, it is clear that if the virtual displacements are only those representing the rigid body degrees of freedom then the virtual work is equal to zero. Hence

$$(\mathbf{Q}_v^{(r)})'\delta\mathbf{v} + (\mathbf{Q}_w^{(r)})'\delta\mathbf{w} = 0 \tag{33}$$

Substituting Eqs. (29a), (31), and (32) into (33), and noting that $\mathbf{K}_{vw}^{(r)} = (\mathbf{K}_{wv}^{(r)})'$, it follows that

$$[\mathbf{v}_0'(\mathbf{K}_{wv}' + \mathbf{K}_{vv}'\mathbf{T}) + \mathbf{w}'(\mathbf{T}'\mathbf{K}_{wv}' + \mathbf{K}_{ww}' + \mathbf{T}'\mathbf{K}_{vv}'\mathbf{T} + \mathbf{K}_{wv}\mathbf{T})]\delta\mathbf{w} = 0 \tag{34}$$

where for simplicity the superscripts (r) have been omitted. As Eq. (34) must be valid for any arbitrary values of $\mathbf{v}_0$ and $\mathbf{w}$,

$$\mathbf{K}_{wv}' + \mathbf{K}_{vv}'\mathbf{T} = 0 \tag{35}$$

and

$$\mathbf{T}'\mathbf{K}_{wv}' + \mathbf{K}_{ww}' + \mathbf{T}'\mathbf{K}_{vv}'\mathbf{T} + \mathbf{K}_{wv}\mathbf{T} = 0 \tag{36}$$

The stiffness submatrices $\mathbf{K}_{wv}$ and $\mathbf{K}_{ww}$ can now be determined from Eqs. (35) and (36). The result is

$$\mathbf{K}_{wv} = -\mathbf{T}'\,\mathbf{K}_{vv} \tag{37}$$

$$\mathbf{K}_{ww} = \mathbf{T}'\,\mathbf{K}_{vv}\mathbf{T} \tag{38}$$

Finally, substitution of Eqs. (37) and (38) into Eq. (29a) leads to

$$\mathbf{K}_b^{(r)} = \left[\begin{array}{c|c} \mathbf{K}_{vv}^{(r)} & -\mathbf{K}_{vv}^{(r)}\mathbf{T}(r) \\ \hline -(\mathbf{T}^{(r)})'\mathbf{K}_{vv}^{(r)} & (\mathbf{T}^{(r)})'\mathbf{K}_{vv}^{(r)}\mathbf{T}^{(r)} \end{array}\right] \tag{39}$$

(5) General Solution for Boundary Displacements: Substructure Relaxation

Having determined the boundary stiffnesses $\mathbf{K}_b^{(r)}$ and the reactions $\mathbf{R}_b^{(r)}$ due to specified interior loading, all boundaries are then relaxed simultaneously with the exception of a number of selected displacements serving to form a reference datum for the complete structure. When the boundaries are relaxed, the boundary reactions and any external forces applied on the boundaries will not be in balance and therefore the boundary relaxation will induce boundary displacements of such magnitude so as to satisfy equilibrium at each joint on the boundary. To calculate these boundary displacements, the complete structure can be regarded as an assembly of substructures subjected to external loading

$$\mathbf{Q}_b = -\sum_r \mathbf{R}_b^{(r)} + \mathbf{P}_b \tag{40}$$

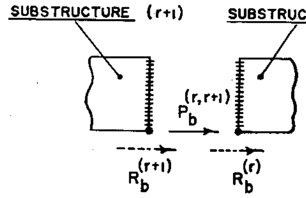

FIG. 2. Joint loads before boundary relaxation.

where the summation implies here addition of the corresponding boundary reactions for boundaries fixed, while $\mathbf{P}_b$ is the loading matrix for external forces applied on the boundaries; the negative sign with $\mathbf{R}_b^{(r)}$ is used to change the boundary reactions into externally applied forces as indicated by Eq. (13). In Fig. 2 a typical joint on the common boundary between substructures (r) and $(r + 1)$ is shown. Here a typical resultant boundary load $Q_b^{(r,r+1)}$ is given by

$$Q_b^{(r,r+1)} = -R_b^{(r)} - R_b^{(r+1)} + P_b^{(r,r+1)} \tag{41}$$

The equations of equilibrium in terms of boundary displacements for the complete structure can now be written as

$$\mathbf{K}_b\,\mathbf{U}_b = \mathbf{Q}_b \tag{42}$$

where $\mathbf{K}_b$ is obtained by placing the submatrices $\mathbf{K}_b^{(r)}$ in their correct positions in the larger framework of the boundary stiffness matrix for the complete structure, and summing all the overlapping terms. Elimination of a sufficient number of displacements to restrain rigid body degrees of freedom for the complete structure ensures that the matrix $\mathbf{K}_b$ is nonsingular, and therefore the boundary displacements $\mathbf{U}_b$ can be determined from

$$\mathbf{U}_b = \mathbf{K}_b^{-1}\,\mathbf{Q}_b \tag{43}$$

Before proceeding to the analysis of loads and displacements on separate substructures, the displacement matrix $\mathbf{U}_b$ must be expanded into a column of substructure displacements $\mathbf{u}_b^{(r)}$, in the exact order in which they appear in Eq. (25a). This can be obtained by a simple matrix transformation:

$$\{\mathbf{u}_b^{(1)}\ \mathbf{u}_b^{(2)}\ \ldots\ \mathbf{u}_b^{(r)}\ \ldots\} = \mathbf{a}_b\,\mathbf{U}_b \tag{44}$$

where the matrix $\mathbf{a}_b$ is of the same type as the transformation matrix $\mathbf{a}$ in Eq. (19).

When the substructure stiffness matrices $\mathbf{K}_b^{(r)}$ are assembled into the larger stiffness matrix $\mathbf{K}_b$ for the complete structure, their relative positions in this larger matrix depend on the sequence in which the individual boundary displacements are selected in Eq. (42). Since some of the substructures will not be physically connected this means that their coupling stiffness matrices will be equal to zero. As the coupling matrices occur only on substructures which have common boundaries, it is therefore advantageous, when selecting a numbering system for substructures and displacements, to ensure that the component submatrices of $\mathbf{K}_b$ would occur around the principal diagonal, forming a band matrix. This arrangement may result in a considerable saving in the computing time, if special inversion programs for band matrices (also known as continuant matrices) are used to determine $\mathbf{K}_b^{-1}$.[17]

A typical substructure arrangement for a delta wing aircraft is shown in Fig. 3; this arrangement results in a quintuple band matrix for $\mathbf{K}_b$ as shown by Eq. (45)

$$\mathbf{K}_b = \begin{bmatrix} [1,1] & [1,2] & 0 & 0 & 0 & 0 & 0 \\ [2,1] & [2,2] & [2,3] & [2,4] & 0 & 0 & 0 \\ 0 & [3,2] & [3,3] & [3,4] & [3,5] & 0 & 0 \\ 0 & [4,2] & [4,3] & [4,4] & [4,5] & 0 & 0 \\ 0 & 0 & [5,3] & [5,4] & [5,5] & [5,6] & [5,7] \\ 0 & 0 & 0 & 0 & [6,5] & [6,6] & 0 \\ 0 & 0 & 0 & 0 & [7,5] & 0 & [7,7] \end{bmatrix} \tag{45}$$

where the row and column numbers denote the substructure numbers. For a high aspect ratio wing aircraft, the substructure boundaries can be selected in such a way as to ensure that each substructure would have common boundaries with not more than two adjacent substructures. A typical arrangement for these cases is shown in Fig. 4. Here even a greater economy of computing effort can be achieved since the boundary stiffness matrix $\mathbf{K}_b$ would result in a triple band matrix as shown by Eq. (46).

$$\mathbf{K}_b = \begin{bmatrix} [1,1] & [1,2] & 0 & 0 & 0 \\ [2,1] & [2,2] & [2,3] & 0 & 0 \\ 0 & [3,2] & [3,3] & [3,4] & 0 \\ 0 & 0 & [4,3] & [4,4] & [4,5] \\ 0 & 0 & 0 & [5,4] & [5,5] \end{bmatrix} \tag{46}$$

(6) The Substructure Displacements and Forces: Boundaries Relaxed

Having determined the boundary displacements on each substructure from Eq. (44), the substructures can be analyzed separately under the external loading $\mathbf{P}_i^{(r)}$ together with known boundary displacements $\mathbf{u}_b^{(r)}$. From Eq. (25a) it follows that the substructure interior displacements $\mathbf{u}_i^{(r)}$ due to the forces $\mathbf{P}_i^{(r)}$ and boundary displacements $\mathbf{u}_b^{(r)}$ are given by

$$\mathbf{u}_i^{(r)} = (\mathbf{K}_{ii}^{(r)})^{-1}\,\mathbf{P}_i^{(r)} - (\mathbf{K}_{ii}^{(r)})^{-1}\,\mathbf{K}_{ib}^{(r)}\mathbf{u}_b^{(r)} \tag{47}$$

Whence

$$\begin{bmatrix} \mathbf{u}_b^{(r)} \\ \hdashline \mathbf{u}_i^{(r)} \end{bmatrix}_{\substack{\text{boundaries} \\ \text{relaxed}}} = \begin{bmatrix} 0 \\ \hdashline \mathbf{u}_i^{(r)} \end{bmatrix}_{\substack{\text{boundaries} \\ \text{fixed}}} + \begin{bmatrix} \mathbf{I} \\ \hdashline -(\mathbf{K}_{ii}^{(r)})^{-1}\mathbf{K}_{ib} \end{bmatrix} \mathbf{u}_b^{(r)} \tag{48}$$

or

$$\begin{bmatrix} \mathbf{U}^{(r)} \end{bmatrix}_{\substack{\text{boundaries} \\ \text{relaxed}}} = \begin{bmatrix} \mathbf{U}^{(r)} \end{bmatrix}_{\substack{\text{boundaries} \\ \text{fixed}}} + \begin{bmatrix} \text{displacements} \\ \text{due to boundary} \\ \text{relaxation} \end{bmatrix} \tag{48a}$$

The substructure displacements obtained from Eq. (48) are then used in Eq. (14) to determine the element forces.

A schematic flow diagram for the complete analysis by the method of substructures is shown in Fig. 5, where for simplicity only the main steps in the computation have been indicated. The diagram illustrates how the individual substructure analyses are assembled together to form boundary stiffness and boundary force matrices for the complete structure followed by the calculation of substructure boundary displacements, which are subsequently used to determine displacements and forces in each substructure independently from one another.

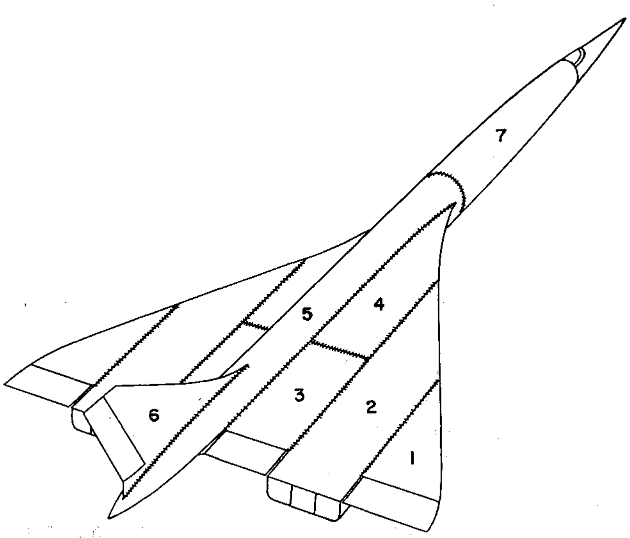

Fig. 3. Typical substructure arrangement for delta aircraft.

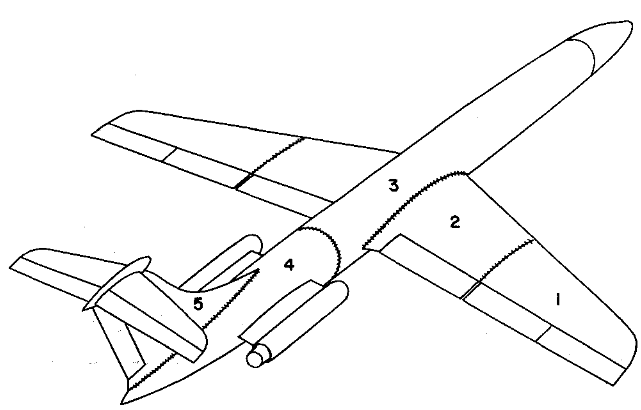

Fig. 4. Typical substructure arrangement for conventional aircraft.

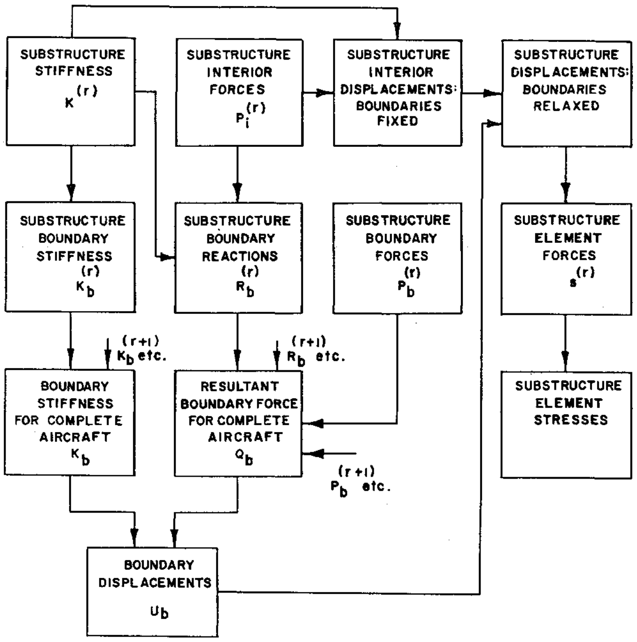

FIG. 5. Flow diagram of analysis.

The general theory for the substructure method of structural analysis has been developed here only on the basis of algebraic symbolism of various matrix operations; related questions on actual programming and computation have not been discussed since these depend on the type of the digital computer used.

(7) Conclusions

From past experience with the analysis of aircraft structures, it is evident that some form of structural partitioning is usually necessary either because different methods of analysis are used on different structural components or because of the limitations imposed by the capacity of digital computers. Even when the next generation of faster and larger digital computers becomes a well-established tool for the analysis of aircraft structures, it seems rather doubtful, because of the large number of unknowns, that the substructure displacement method of analysis would be wholly superseded by an overall analysis carried out on the complete structure. It must also be emphasized that there may be cases where the coupling between various substructures could be achieved more conveniently through the interaction redundant systems, rather than through the matching of displacements. This would, of course, be a natural choice if the matrix force method was used throughout.

Computer programs for the analysis of structures are available[18–20] for both the displacement and force methods which are general with respect to the type of built-up structures, type of structural elements, and their orientation in space. These programs can easily be adapted to the substructure method of analysis outlined in this paper. The main advantage of this method lies in a very compact formulation of the analysis and in the attendant reduction of computing time which can be further reduced by judicious partitioning into substructures. The method does, however, have the disadvantage that the individual substructure analysis results deviate very much from the final result due to boundary fixing, making them in general unsuitable for preliminary use. Only after simultaneous relaxation of all substructure boundaries is a useful result obtained.

Appendix

Derivation of the Transformation Matrix $T^{(r)}$

An equivalent form of Eq. (31) can be written in terms of three orthogonal rigid body translations w_x, w_y, and w_z of a point k on the boundary, and three rigid body rotations θ_x, θ_y, and θ_z about the w_x, w_y, and w_z displacement vectors respectively; therefore omitting for simplicity the superscripts (r) referring to the rth substructure, the equivalent form of Eq. (31) may be expressed as

$$\begin{bmatrix} \mathbf{v} \\ \hline \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0 \\ \hline \mathbf{0} \end{bmatrix} + \left[\begin{array}{c|c} \mathbf{T}_{vw} & \mathbf{T}_{v\theta} \\ \hline \mathbf{T}_{ww} & \mathbf{T}_{w\theta} \end{array}\right] \begin{bmatrix} w_x \\ w_y \\ w_z \\ \hline \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} \tag{49}$$

where the significance of the transformation submatrices $\mathbf{T}_{vw}$, etc. becomes obvious after multiplication by the column matrix $\{w_x\ w_y\ w_z\ \theta_x\ \theta_y\ \theta_z\}$. Assuming now that $w_x = w_1$, $w_y = w_2$, and $w_z = w_3$, where w_1, w_2, and w_3 are three datum displacements of the point k in the 0x, 0y, and 0z directions respectively, the equation for $\mathbf{w}$ from Eq. (49) becomes

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \mathbf{T}_{ww} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \mathbf{T}_{w\theta} \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} \tag{50}$$

The transformation submatrices $\mathbf{T}_{ww}$ and $\mathbf{T}_{w\theta}$ can be partitioned further into

$$\mathbf{T}_{ww} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} \quad \text{and} \quad \mathbf{T}_{w\theta} = \begin{bmatrix} \mathbf{C} \\ \mathbf{D} \end{bmatrix} \tag{51}$$

which, when substituted into Eq. (50), lead to two simultaneous equations

$$\mathbf{C} \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = (\mathbf{I} - \mathbf{A}) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \tag{52}$$

$$\mathbf{D} \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \begin{bmatrix} w_4 \\ w_5 \\ w_6 \end{bmatrix} - \mathbf{B} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \tag{53}$$

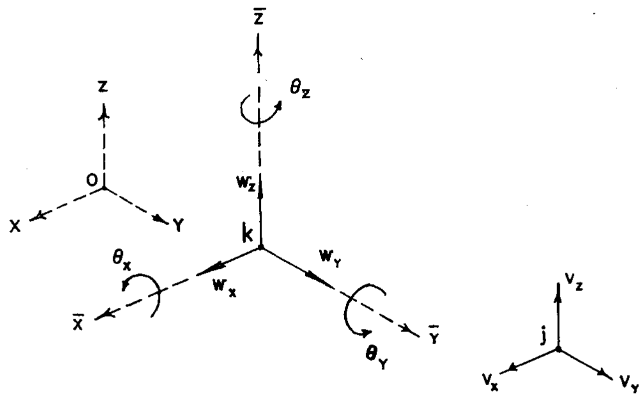

FIG. 6. Sign convention for boundary displacements and rotations.

As Eq. (52) must be satisfied for all values of $\{\theta_x\ \theta_y\ \theta_z\}$ and $\{w_1\ w_2\ w_3\}$ it follows immediately that

$$\mathbf{C} = \mathbf{0} \quad \text{and} \quad \mathbf{A} = \mathbf{I} \tag{54}$$

while the solution of Eq. (53) becomes

$$\begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \mathbf{D}^{-1} \begin{bmatrix} w_4 \\ w_5 \\ w_6 \end{bmatrix} - \mathbf{D}^{-1}\mathbf{B} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \tag{55}$$

The column matrix $\{w_x\ w_y\ w_z\ \theta_x\ \theta_y\ \theta_z\}$ in Eq. (49) can now be related to the six datum displacements $w_1 \ldots w_6$; thus using Eq. (55) the following relationship is derived

$$\begin{bmatrix} w_x \\ w_y \\ w_z \\ \hline \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{O} \\ \hline -\mathbf{D}^{-1}\mathbf{B} & \mathbf{D}^{-1} \end{array}\right] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \hline w_4 \\ w_5 \\ w_6 \end{bmatrix} \tag{56}$$

which can be substituted into the second term on the right-hand side of Eq. (49) so that

$$\left[\begin{array}{c|c} \mathbf{T}_{vw} & \mathbf{T}_{v\theta} \\ \hline \mathbf{T}_{ww} & \mathbf{T}_{w\theta} \end{array}\right] \begin{bmatrix} w_x \\ w_y \\ w_z \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \left[\begin{array}{c|c} (\mathbf{T}_{vw} - \mathbf{T}_{v\theta}\mathbf{D}^{-1}\mathbf{B}) & \mathbf{T}_{v\theta}\mathbf{D}^{-1} \\ \hline (\mathbf{T}_{ww} - \mathbf{T}_{w\theta}\mathbf{D}^{-1}\mathbf{B}) & \mathbf{T}_{w\theta}\mathbf{D}^{-1} \end{array}\right] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \hline w_4 \\ w_5 \\ w_6 \end{bmatrix} =$$

$$\left[\begin{array}{c|c} (\mathbf{T}_{vw} - \mathbf{T}_{v\theta}\mathbf{D}^{-1}\mathbf{B}) & \mathbf{T}_{v\theta}\mathbf{D}^{-1} \\ \hline \multicolumn{2}{c}{\mathbf{I}} \end{array}\right] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \hline w_4 \\ w_5 \\ w_6 \end{bmatrix} \tag{57}$$

where

$$[(\mathbf{T}_{ww} - \mathbf{T}_{w\theta}\mathbf{D}^{-1}\mathbf{B}) \mid \mathbf{T}_{w\theta}\mathbf{D}^{-1}] = \mathbf{I} \tag{58}$$

may be easily verified by substituting Eqs. (51) for $\mathbf{T}_{ww}$ and $\mathbf{T}_{w\theta}$ with $\mathbf{A} = \mathbf{I}$ and $\mathbf{C} = \mathbf{0}$.

Substituting now Eq. (57) into Eq. (49), and then equating Eqs. (31) and (49), it can be shown that the transformation matrix $\mathbf{T}$ is given by

$$\mathbf{T} = [(\mathbf{T}_{vw} - \mathbf{T}_{v\theta}\mathbf{D}^{-1}\mathbf{B}) \mid \mathbf{T}_{v\theta}\mathbf{D}^{-1}] \tag{59}$$

where the matrices $\mathbf{T}_{vw}$, $\mathbf{T}_{v\theta}$, $\mathbf{B}$, and $\mathbf{D}$ are determined from the topology of structural displacements.

The function of the matrix $\mathbf{T}_{vw}$ is to relate the substructure boundary displacements $\mathbf{v}$ to the rigid body translations w_1, w_2, and w_3. Thus, the matrix $\mathbf{T}_{vw}$ can be expressed as

$$\mathbf{T}_{vw} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \lambda_{j1} & \lambda_{j2} & \lambda_{j3} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \tag{60}$$

where λ_{jk} denotes the direction cosine of the angle between the substructure boundary displacement vector v_j, other than the datum displacement, and the rigid body translation w_k, where $k = 1, 2$, and 3. Since the displacements v_j and w_k are related to a common set of axes, it is clear that the direction cosines λ_{jk} are either equal to one or to zero, depending on whether the corresponding directions coincide or are mutually orthogonal. If the displacements $\mathbf{v}$ are arranged into a column matrix $\{v_{1x}\ v_{1y}\ v_{1z}\ v_{2x}\ v_{2y}\ v_{2z} \ldots v_{jx}\ v_{jy}\ v_{jz} \ldots\}$, where the first suffix denotes the node (joint) number, while the second suffix denotes the direction of the displacement, then the matrix $\mathbf{T}_{vw}$ becomes a column matrix

$$\mathbf{T}_{vw} = \{\mathbf{I}_3\ \ \mathbf{I}_3 \ldots \mathbf{I}_3 \ldots\} \tag{60a}$$

where $\mathbf{I}_3$ is a unit matrix of order 3×3.

The matrix $\mathbf{T}_{v\theta}$ relates the substructure boundary displacements $\mathbf{v}$ and the rigid body rotations θ_x, θ_y, and θ_z about the displacement vectors w_1, w_2, and w_3. Thus taking three displacements v_{jx}, v_{jy}, and v_{jz} at the joint j, whose coordinates are x_j, y_j, and z_j, it follows from Fig. 6 that

$$\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \mathbf{v}_{jx} \\ \mathbf{v}_{jy} \\ \mathbf{v}_{jz} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \mathbf{T}_{v\theta} \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \bar{z}_j & -\bar{y}_j \\ -\bar{z}_j & 0 & \bar{x}_j \\ \bar{y}_j & -\bar{x}_j & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} \tag{61}$$

where

$$\begin{aligned} \bar{x}_j &= x_j - x_k \\ \bar{y}_j &= y_j - y_k \\ \bar{z}_j &= z_j - z_k \end{aligned} \tag{62}$$

Similarly the matrices $\mathbf{B}$ and $\mathbf{D}$ relate the remaining three datum displacements w_4, w_5, and w_6 to the three rigid body translations w_1, w_2, and w_3, and the three rotations θ_x, θ_y, and θ_z, respectively. Whence, it is clear that

$$\mathbf{B} = \begin{bmatrix} \lambda_{41} & \lambda_{42} & \lambda_{43} \\ \lambda_{51} & \lambda_{52} & \lambda_{53} \\ \lambda_{61} & \lambda_{62} & \lambda_{63} \end{bmatrix} \tag{63}$$

is the matrix of direction cosines of the angle between the substructure datum displacement vectors w_4, w_5, and w_6 and the rigid body translation vectors w_1, w_2, and w_3. Since w_4, w_5, and w_6 are taken in the x, y, and z-directions, respectively, it follows therefore that

$$\mathbf{B} = \mathbf{I} \tag{63a}$$

Finally, for the w_4, w_5, and w_6 displacements, as specified above,

$$\mathbf{D} = \begin{bmatrix} 0 & \bar{z}_4 & -\bar{y}_4 \\ -\bar{z}_5 & 0 & \bar{x}_5 \\ \bar{y}_6 & -\bar{x}_6 & 0 \end{bmatrix} \tag{64}$$

and

$$\mathbf{D}^{-1} = \frac{1}{|D|} \begin{bmatrix} \bar{x}_5\bar{x}_6 & \bar{y}_4\bar{x}_6 & \bar{z}_4\bar{x}_5 \\ \bar{x}_5\bar{y}_6 & \bar{y}_4\bar{y}_6 & \bar{y}_4\bar{z}_5 \\ \bar{z}_5\bar{x}_6 & \bar{z}_4\bar{y}_6 & \bar{z}_4\bar{z}_5 \end{bmatrix} \tag{65}$$

where $\bar{z}_4, \ldots,$ etc. are calculated from Eq. (62). It should, however, be observed that since $\mathbf{D}^{-1}$ is required in Eq. (55), the matrix $\mathbf{D}$ must be nonsingular and this condition is expressed as

$$|D| = \bar{z}_4\bar{x}_5\bar{y}_6 - \bar{y}_4\bar{z}_5\bar{x}_6 \neq 0 \tag{66}$$

The above equation should always be used to check that the choice of the displacements w_4, w_5, and w_6 is in fact a correct one.

It can be shown, using analytical geometry, that $|D| = 0$ represents the condition for the lines of action of the constraining displacement vectors w_4, w_5, and w_6 to pass through a straight line in space drawn from the datum point k. This implies that the three displacement vectors w_4, w_5, and w_6 would then have a zero moment about this line and consequently it would not be possible to restrain rotation about this line with force reactions in the direction of the specified displacements.

It is also recommended that whenever six displacements, say $U_1, \ldots, U_6$, are eliminated from the complete stiffness matrix $\mathbf{K}$ in order to make it nonsingular, and the three orthogonal displacements U_1, U_2, and U_3 refer to a common point, then the coordinates specifying the location of the remaining three displacement vectors U_4, U_5, and U_6 should be checked in Eq. (66). This would ensure that all rigid body degrees of freedom are eliminated from the reduced stiffness matrix used for the calculation of structural displacements.

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